SIMULATION OF A TRANSIENT HEAT-CONDUCTION PROCESS

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A method is described of simulating a transient heat-conduction process on a general-purpose analog computer. For illustration, the problem of optimally heating a polymer film has been solved by this method.

The transmission of thermal energy through various kinds of walls is in many industrial applications effected by a transient mode of heat conduction. Thus, there arise problems of optimization.

One very often encounters problems in temperature regulation, which in many ways determines the product quality, and reduces them to that of simulating the object through which transient heat conduction takes place while it interacts with the control device. The simulation process reveals the optimum regulation modes and determines the optimum parameters of the automatic control system (ACS). Such problems are solved with the aid of analog computers (AC) [1, 2].

Analog-computer simulation of general transient heat-conduction processes and the necessity of considering a three-dimensional temperature field give rise to various difficulties [3] involved with the analog representation of partial differential equations with variable boundary conditions and with the resulting transcendental transfer functions dependent on the boundary conditions of a given problem. In this study we will consider one possible method of overcoming these difficulties.

We consider the problem of transient heat conduction through an infinitely large plane wall:

$$\frac{\partial\theta(\xi;\ \mathrm{Fo})}{\partial\,\mathrm{Fo}} = \frac{\partial^2\theta(\xi;\ \mathrm{Fo})}{\partial\xi^2}, \tag{1}$$

$$\mathrm{Fo} > 0;\ 0 \leqslant \xi \leqslant 1.$$

with the initial and the boundary conditions

$$\theta(\xi;\ 0) = 0, \tag{2}$$

$$\frac{\partial\theta(0;\ \mathrm{Fo})}{\partial\xi} + q_1(\mathrm{Fo}) = 0, \tag{3}$$

$$\frac{\partial\theta(1;\ \mathrm{Fo})}{\partial\xi} - q_2(\mathrm{Fo}) = 0. \tag{4}$$

Quantities q_1 and q_2 in Eqs. (3) and (4) are the referred thermal fluxes on the wall surfaces:

$$q_1(\mathrm{Fo}) = \frac{l}{\lambda} q_{11}(\mathrm{Fo}) - \mathrm{Bi}_1[\theta(0;\ \mathrm{Fo}) - \theta_1(\mathrm{Fo})], \tag{5}$$

$$q_2(\mathrm{Fo}) = \frac{l}{\lambda} q_{21}(\mathrm{Fo}) - \mathrm{Bi}_2[\theta(1;\ \mathrm{Fo}) - \theta_2(\mathrm{Fo})]. \tag{6}$$

We apply the Laplace transformation to Eqs. (1)-(6) with respect to Fo [4]. The transform functions will be denoted by capital letters. Solving the transform equations with the constraints will yield

$$\Theta(\xi;\ p) = W_1(\xi;\ p)Q_1(p) + W_2(\xi;\ p)Q_2(p), \tag{7}$$

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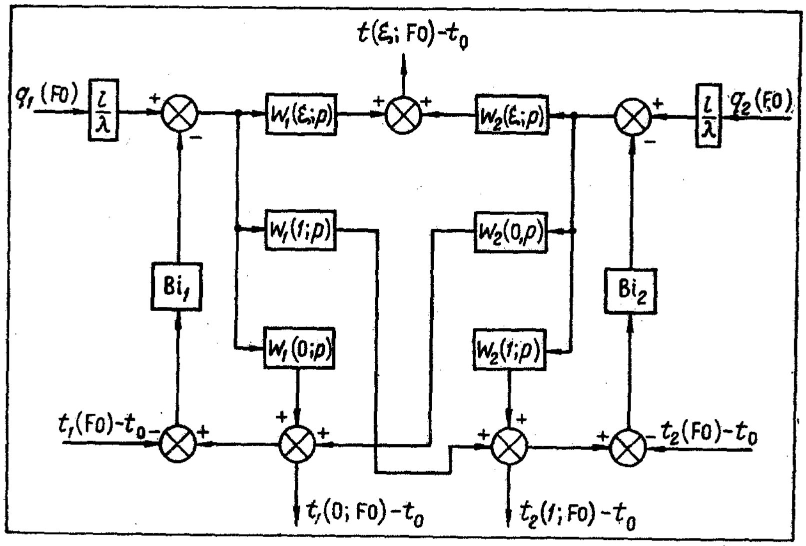


Fig. 1. Structural diagram of a model.

$$W_1(\xi;\ p)=\frac{\mathrm{ch}\,[\sqrt{p}\,(1-\xi)]}{\sqrt{p}\,\mathrm{sh}\sqrt{p}}\ ; \tag{8}$$

$$W_2(\xi;\ p)=\frac{\mathrm{ch}\,(\sqrt{p}\,\xi)}{\sqrt{p}\,\mathrm{sh}\sqrt{p}}\ . \tag{9}$$

Let us now examine the structure of functions Q_1 and Q_2

$$Q_1(p)=\frac{l}{\lambda}Q_{11}(p)-\mathrm{Bi}_1[\Theta(0;\ p)-\Theta_1(p)]. \tag{10}$$

With (7) taken into account, we have

$$Q_1(p)=\frac{l}{\lambda}Q_{11}(p)-\mathrm{Bi}_1[W_1(0;\ p)Q_1(p)+W_2(0;\ p)Q_2(p)-\Theta_1(p)]. \tag{11}$$

For Q_2 we have, analogously,

$$Q_2(p)=\frac{l}{\lambda}Q_{21}(p)-\mathrm{Bi}_2[W_1(1;\ p)Q_1(p)+W_2(1;\ p)Q_2(p)-\Theta_2(p)]. \tag{12}$$

On the basis of Eqs. (7), (11), and (12), we design a structural diagram (Fig. 1) of a model which combines operators and transform functions in the form (8) and (9).

The form of the transfer functions does not depend on the specific conditions of the problem and is determined only by the wall geometry.

The structure of a model depends on the boundary conditions. Boundary conditions of the third kind result in feedback coupling at the surfaces and in transfer functions representing the interaction between processes at these surfaces.

In order to make an analog-computer simulation of the process feasible, we have expanded the transcendental transfer functions into series with respect to their poles:

$$W_1(\xi;\ p)=\frac{2}{p}+2\sum_{n=1}^{\infty}\frac{\frac{1}{n^2\pi^2}\cos n\pi\xi}{\frac{1}{n^2\pi^2}\cdot p+1}\ , \tag{13}$$

$$W_2(\xi;\ p)=\frac{2}{p}+2\sum_{n=1}^{\infty}(-1)^n\frac{\frac{1}{n^2\pi^2}\cos n\pi\xi.}{\frac{1}{n^2\pi^2}p+1}\ . \tag{14}$$

The gain coefficients in the expansion terms represent a fast decreasing (proportionally to n^2) alternating series, which means that little accuracy is lost by terminating it.

For problems with $Bi \leq 1$ it is sufficient to terminate the series after $n = 1$; for problems where $1 < Bi < \infty$, $n = 3$ will yield an accuracy of 5%.

If it becomes necessary to improve the accuracy of the solution, then one includes more terms of the series in the calculation. It is noteworthy that the time constants in the first-order terms of these expansions are independent of ξ. As a consequence, switching from one section (of ξ-values) to another involves only a change of respective gain coefficients. The gain coefficients of W_1 and W_2 differ in sign only. All this simplifies the analog-computer design of a model.

Thus, four integrating networks are needed for the representation of expansion terms, and 2n networks are needed for the representation of n terms. The correctness of the analog-computer model and the accuracy of the obtained solution are checked against the solution to a control problem with boundary conditions of the third kind and by matching the results against the curves given in Lykov's book [4].

An analogous problem is solved for a cylindrical wall:

$$\frac{\partial\theta(\xi;\ Fo)}{\partial Fo} = \frac{\partial^2\theta(\xi;\ Fo)}{\partial\xi^2} + \frac{1}{\xi}\cdot\frac{\partial\theta(\xi;\ Fo)}{\partial\xi}, \tag{15}$$

$$Fo > 0;\ \xi_1 \leqslant \xi \leqslant \xi_2,$$

$$\frac{\partial\theta(\xi_1;\ Fo)}{\partial\xi} + q_1(Fo) = 0, \tag{16}$$

$$\frac{\partial\theta(\xi_2;\ Fo)}{\partial\xi} - q_2(Fo) = 0, \tag{17}$$

$$q_1(Fo) = \frac{r_1}{\lambda} q_{11}(Fo) - Bi_1[\theta(\xi_1;\ Fo) - \theta_1(Fo)], \tag{18}$$

$$q_2(Fo) = \frac{r_1}{\lambda} q_{21}(Fo) - Bi_2[\theta(\xi_2;\ Fo) - \theta_2(Fo)]. \tag{19}$$

The structural schematic diagram of this model is analogous to the previous one. The basic transcendental transfer functions and their series expansions are shown in Table 1.

As an example, we will now consider the problem of optimally heating a polymer film during its orientation. Prior to the orientation process, such a film is heated by passing it over a set of rollers sequentially at a constant velocity (Fig. 2a).

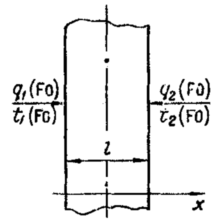
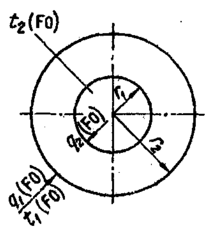
The number of rollers, their diameter, the contact angle, the velocity, and the film thickness are usually known or are determined by the machine design parameters and by the requirements of the specific technological process. The conditions of heat transfer between roller and film are also usually known. It is required to determine the temperature of the rollers, within imposed limits, which will ensure the minimum sum of all their temperatures at a given heating uniformity. This is equivalent to minimizing the energy losses and the degree of crystallization due to heating, the latter causing a deterioration of the film quality.

The problem here will be formulated analytically. The object is described by Eq. (1) with the boundary conditions

$$\begin{gathered}\frac{\partial\theta(0;\ Fo)}{\partial\xi} - \gamma_1(Fo)\,Bi[\theta_i(Fo) - \theta(0;\ Fo)] = 0,\\ \frac{\partial\theta(1;\ Fo)}{\partial\xi} + \gamma_2(Fo)\,Bi[\theta_i(Fo) - \theta(1;\ Fo)] = 0.\end{gathered} \tag{20}$$

Here γ_1 and γ_2 are step functions of the Fourier number, equal to 1 or 0 depending on the number of rollers with which the film is in contact at a given instant of time: $\gamma_1 \neq \gamma_2$. A switching of γ_1 and γ_2 corresponds to a pass of the film from one roller to another. The roller temperature θ_i is then also switched.

TABLE 1. Basic Transfer Functions

Wall type	Schematic diagram of the physical model	Basic transfer functions	Series expansion of basic transfer function
Plane		$W_1(\xi;\ p) = \frac{\mathrm{ch}\,[\sqrt{p}\,(1-\xi)]}{\sqrt{p}\,\mathrm{sh}\,\sqrt{p}}$; $W_2(\xi;\ p) = \frac{\mathrm{ch}\,[\sqrt{p}\,\xi]}{\sqrt{p}\,\mathrm{sh}\,\sqrt{p}}$; $\xi = \frac{x}{l};\ 0 \leqslant \xi \leqslant 1;$ $\mathrm{Fo} = \frac{a\tau}{l^2}$	$W_1(\xi;\ p) = \frac{2}{p} + 2\sum_{n=1}^{\infty}\frac{\frac{1}{n^2\pi^2}\cos n\pi\xi}{\frac{1}{n^2\pi^2}p+1}$; $W_2(\xi;\ p) = \frac{2}{p} + 2\sum_{n=1}^{\infty}(-1)^n\frac{\frac{1}{n^2\pi^2}\cos n\pi\xi}{\frac{1}{n^2\pi^2}p+1}$; $n = 1,\ 2,\ 3\ldots$
Cylindrical		$W_1(\xi;\ p) = \frac{N_1(i\sqrt{p}\,\xi_2)\,I_0(i\sqrt{p}\,\xi) - I_1(i\sqrt{p}\,\xi_2)N_0(i\sqrt{p}\,\xi)}{i\sqrt{p}\,[I_1(i\sqrt{p})N_1(i\sqrt{p}\,\xi_2) - I_1(i\sqrt{p}\,\xi_2)N_1(i\sqrt{p})]}$; $W_2(\xi;\ p) = \frac{N_1(i\sqrt{p})I_0(i\sqrt{p}\,\xi) - I_1(i\sqrt{p})N_0(i\sqrt{p}\,\xi)}{i\sqrt{p}\,[I_1(i\sqrt{p})N_1(i\sqrt{p}\,\xi_2) - I_1(i\sqrt{\rho\xi_2})N_1(i\sqrt{p})]}$; $\xi = \frac{r}{r_1};\ 1 \leqslant \xi \leqslant \frac{r_2}{r_1}$; $\mathrm{Fo} = \frac{a\tau}{r_1^2}$	$W_1(\xi;\ p) = \frac{2}{[(\xi_2)^2-1]p} + 2\sum_{n=1}^{\infty}\frac{A_{1n}}{T_np+1}$; $W_2(\xi;\ p) = \frac{2\xi_2}{[(\xi_2)^2-1]\,p} + 2\sum_{n=1}^{\infty}\frac{A_{2n}}{T_np+1}$; $A_{1n} = \frac{N_1(\mu_n\xi_2)\,I_0(\mu_n\xi) - I_1(\mu_n\xi_2)\,N_0(\mu_n\xi)}{\mu_n^2B_n}$; $A_{2n} = \frac{N_1(\mu_n)\,I_0(\mu_n\xi) - I_1(\mu_n)\,N_0(\mu_n\xi)}{\mu_n^2B_n}$; $B_n = \{\xi_2\,[I_0(\mu_n\xi_2)\,N_1(\mu_n) - I_1(\mu_n)\,N_0(\mu_n\xi_2)] -$ $- [I_0(\mu_n)\,N_1(\mu_n\xi_2) - I_1(\mu_n\xi_2)\,N_0(\mu_n)]\}$; $n = T\frac{1}{\mu_n^2}$.

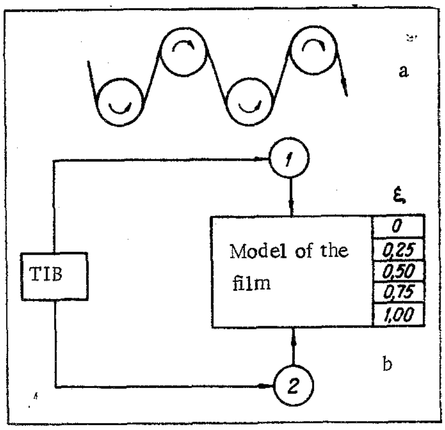


Fig. 2

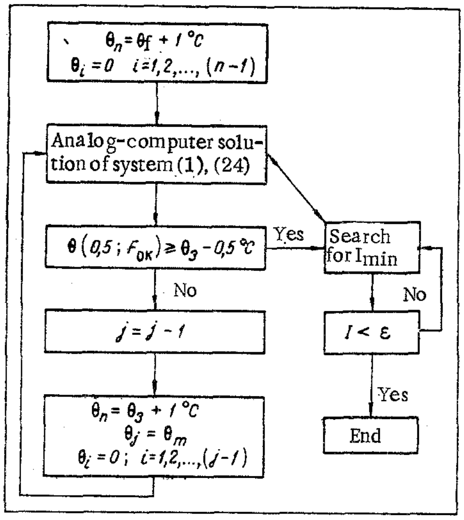


Fig. 3

Fig. 2. a) Schematic diagram of film passage over rollers; b) schematic diagram of problem simulation on an analog computer.

Fig. 3. Schematic block diagram of the algorithm.

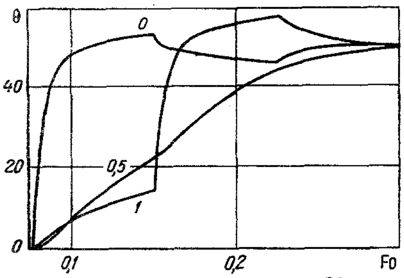


Fig. 4. Optimum heating of a film. Numbers at the curves refer to the respective values of $\xi \cdot \theta$, °C.

According to the number of rollers, there are n time intervals determined by the roller radius, the contact angle with the film, and the velocity of film travel:

$$\mathrm{Fo}_i = \frac{a\tau_i}{\delta^2}, \qquad \tau_i = \frac{\varphi R}{v}. \tag{21}$$

It is required to determine now

$$\min \sum_i \theta_i \tag{22}$$

under conditions of

$$0 \leqslant \theta_i \leqslant \theta_m. \tag{23}$$

Here θ_m is the maximum allowable roller temperature, above which the film may stick to the roller.

$$I = \int_0^1 [\theta(\xi;\ \mathrm{Fo}_k) - \theta_f]^2 d\xi \leqslant \varepsilon. \tag{24}$$

Here ε is a small quantity which determines the allowable temperature deviation within a film section upon exit from the heating zone at the final instant of time Fo_k.

The problem was solved on a model MN-14 analog computer (Fig. 2b). The object, described by Eqs. (1), (20), and (21), was simulated according to the method shown here. The time-interval unit (TIB) of the analog computer switched the boundary conditions 1 and 2 at the instants of time Fo_i defined by Eq. (21). The functional (24) was replaced by a sum of terms at five points $\xi = 0$, 0.25, 0.50, 0.75, 1.0:

$$I = \sum_{m=1}^{5} [\theta(\xi_m;\ \mathrm{Fo}_k) - \theta_f]^2. \tag{25}$$

The problem was reduced to finding the conditional minimum of a multivariable function by methods of nonlinear programming as, for instance, the gradient methods [5]. Tracking the minimum (22) was replaced by tracking the minimum of sum (25). In order to ensure finding a solution to the given problem, however, the computation steps following the determination of the gradient components

$$\Delta_i = \frac{\Delta I}{\Delta \theta_i} \tag{26}$$

TABLE 2. Optimum Heating Mode

θ_1, °C	θ_2, °C	θ_3, °C	θ_4, °C	$\theta(\xi;\ Fo_K)$ for ξ					I	$\Sigma\theta_i$
				0	0,25	0,5	0,75	1,0		
0	58	60	51	50,5	49,8	49,8	50,3	50,6	0,78	169

were made inversely proportional to the gradient components:

$$\theta_i^k = \theta_i^{k-1} - h\,\frac{1}{\Delta_i}. \tag{27}$$

This made it possible to approach I_{min} with the maximum temperature decrease at those rollers which least affected the value of I. While I_{min} was reached under the limiting conditions of the problem, therefore, the minimum of (22) was reached at the same time. A block diagram of the algorithm is shown in Fig. 3. The search for I_{min} began from the last roller along the film route. A temperature close to the rated film temperature was specified on this roller, namely $\theta_f + 1$°C. A solution for the given conditions was obtained on the analog computer and the temperature at the center of the film section

$$\theta(0.5;\ Fo_k) \geqslant \theta_f - 0.5\,°C \tag{28}$$

was checked. Here ±0.5°C was the permissible temperature deviation from the specified level. When inequality (28) was found to be satisfied, then I_{min} was found by the method (26), (27) and the problem was considered solved. When inequality (28) was found not to be satisfied, then the second roller at temperature θ_m was hooked on and Eqs. (1), (20) were solved on the analog computer. Inequality (28) was checked again and, if it was found not to be satisfied, the next roller at temperature θ_m was hooked on. This procedure was repeated until the addition of another roller for heating the film made inequality (28) hold true. This preliminary search ensured an exit into the zone of I_{min}. The procedure (26), (27) was then used for the final determination of the temperatures of rollers which would yield the solution to the original problem.

For the case of four heating rollers in a machine with a 0.002 m (2000 μ) thick polyethylene-terephthalate film passing at a velocity of 12 m/min over the rollers with a π-radians (180°) contact angle, this method yielded a solution in 16 min. A technician performed the logic operations, changed the temperatures of rollers, and read the output data on a digital voltmeter. The rated film temperature was θ_f = 50°C, θ_m = 60°C, t_0 = 20°C, ε = 1,

$$a = 8 \cdot 10^{-8}\ m^2/sec;\ \lambda = 0.141\ W/m \cdot deg;\ Bi = 16.5;\ R = 0.2\ m.$$

Curves of the film temperature versus the Fourier number are shown in Fig. 4 for ξ = 0, 0.5, and 1.0 with the optimum temperatures of rollers. The temperatures of rollers, the temperatures at five points on a film section at the end of the heating process, the values of I and of $\Sigma\theta_i$ are all given in Table 2.

The described procedure for solving optimization problems in the area of film heating has been introduced in industrial plants for the production of biaxially oriented polymer films and has made it feasible to improve the physicomechanical properties of such films.

NOTATION

$\theta = t - t_0$;
$\theta_1 = t_1 - t_0$;
$\theta_2 = t_2 - t_0$;

t	is the film temperature;
t_0	is the initial film temperature;
t_1, t_2	are the temperatures of the media where Newtonian heat transfer occurs at the respective wall surfaces;
$\xi = x/l$	is the dimensionless coordinate;
x	is the space coordinate;
l	is the wall thickness;
$Fo = a\tau/l^2$	is the Fourier number;
a	is the thermal diffusivity;
τ	is the time coordinate;
λ	is the thermal conductivity;

$Bi = \alpha l/\lambda$ is the Biot number;

α is the heat-transfer coefficient;

p is the Laplace transformation operator; in Eqs. (15), (16), and (17):

$\xi = r/r_1$;

$\xi_1 = 1$;

$\xi_2 = r_2/r_1$;

r_1 is the inside radius of cylindrical wall;

r_2 is the outside radius of cylindrical wall;

$\theta_i = t_i - t_0$;

t_i is the temperature of i-th roller along the film travel contact angle between film and roller;

R is the radius of roller;

v is the velocity of film travel;

$\theta_f = t_f - t_0$ is the rated film temperature;

θ_i^k is the temperature of i-th roller on the k-th step of tracking the minimum;

h is the constant.

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